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RELIABILITY SYSTEMS WITH EXPONENTIAL SYSTEM LIFE AND IFRA COMPO--ETC(U)

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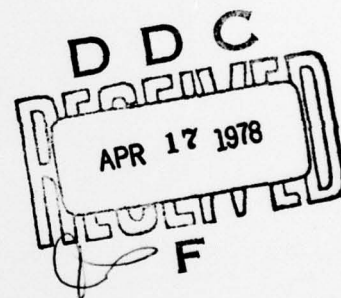
Reliability Systems with Exponential System Life
and IFRA Component Lives

by

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1. Introduction.

Many kinds of reliability systems can be formed using independent components. It turns out, however, that only in very special cases can the system lifetime be exponential if the components have increasing failure rate average (IFRA). In particular, we show (Theorem 2.1) that if a monotonic system formed with independent IFRA components has exponential life, then it must be essentially a series system with exponential components. Results similar to this have been mentioned in Esary, Marshall and Proschan (1970) and Esary and Marshall (1975), but these require some assumptions on the support of the component lifetime distributions. We need no such assumptions in our proof.

In the case of systems whose life is the sum of its component lives we obtain a similar result. Theorem 2.8 states that if the convolution of independent IFRA lifetimes is exponential, then one must be exponential and all others are degenerate at zero. Some related results about parallel systems of two dependent components whose lifetime is exponential are also given in Theorems 2.4 - 2.6 .

In section three we use these results to give counterexamples which show that various definitions of multivariate IFRA distributions are not equivalent.

All definitions and notation follow that of Barlow and Proschan (1975).

2. Results.

(2.1) Theorem. Let F be the life distribution of a monotonic system of order n formed with independent IFRA components having life distributions $F_1(t), \dots, F_n(t)$ respectively. Then there exists a subcollection $1 \leq i_1 < \dots < i_k \leq n$ such that for all $t \geq 0$

$$\bar{F}(t) = \bar{F}_{i_1}(t) \dots \bar{F}_{i_k}(t)$$

and each $F_{i_j}(t)$ is exponential.

Before proving the theorem we introduce a lemma which will be used in the proof of the theorem.

(2.2) Lemma. Let $h(\underline{p})$ be the reliability function of a monotonic system of n independent components and p_i the probability that the i th component is functioning. If there is $0 < \alpha < 1$ such that

$$p_i^\alpha h^\alpha(1_i, \underline{p}) + (1-p_i)^\alpha h^\alpha(0_i, \underline{p}) = h(\underline{p})$$

then at least one of the following three conditions must hold:

$$1) \quad p_i = 0 \quad \text{or} \quad p_i = 1$$

$$2) \quad h(1_i, \underline{p}) = h(0_i, \underline{p})$$

$$3) \quad h(0_i, \underline{p}) = 0.$$

Proof. This follows from the proof of Lemma 2.3, page 84, of Barlow and Proschan (1975) since $f(x) = x^\alpha$ is strictly concave in $x \geq 0$.

Proof of Theorem. Let $\bar{F}(t) = e^{-\lambda t} = h(\bar{F}_1(t), \dots, \bar{F}_n(t))$ where h is the reliability function of a monotonic system ϕ of order n , and $\bar{F}_i(t)$ are IFRA for $i = 1, \dots, n$. Now ϕ has the representation $\phi(\underline{x}) = \min_{1 \leq j \leq p} \max_{i \in K_j} x_i$ where x_i is the state of the i th relevant component and K_j is a min cut set. So if T_1, \dots, T_n are the lifetimes, then

$$e^{-\lambda t} = P\{\min_{j=1, \dots, p} \max_{i \in K_j} T_i > t\}.$$

We shall now show that there is an i such that $0 < \bar{F}_i(t) < 1$ for all $t > 0$. First $e^{-\lambda t} > 0$ for all $t > 0$ gives that

$P\{\max_{i \in K_j} T_i > t\} > 0$ for each $j = 1, \dots, p$. Consequently, for each

$j = 1, \dots, p$, there is some $i_j \in K_j$ such that $P\{T_{i_j} > t\} > 0$ for all

$t > 0$. Now $1 > e^{-\lambda t}$ for all $t > 0$ implies that there is a j_0 such that $1 > P\{\max_{i \in K_{j_0}} T_i > t\} = 1 - \prod_{i \in K_{j_0}} P\{T_i \leq t\}$ and so for all $t > 0$

$P\{T_i > t\} < 1$ for each $i \in K_{j_0}$. Thus for $i_{j_0} \in K_{j_0}$, $0 < \bar{F}_{i_{j_0}}(t) < 1$

for all $t > 0$.

We now use an induction argument on the order of the monotonic system. Clearly the result is true for $n = 1$. Assume it is true for any monotonic system of order less than n and consider a monotonic system of order n . By the pivotal decomposition using the i above

$$e^{-\lambda t} = \bar{F}_1(t) [h(1_1, \bar{F}(t)) - h(0_1, \bar{F}(t))] + h(0_1, \bar{F}(t)),$$

where $h((\cdot)_1, \bar{F}(t)) = h(\bar{F}_1(t), \dots, (\cdot)_1, \dots, \bar{F}_n(t))$. Now by the

lemma since $0 < \bar{F}_1(t) < 1$ for all $t > 0$, then either

$h(1_1, \bar{F}(t)) = h(0_1, \bar{F}(t))$ or $h(0_1, \bar{F}(t)) = 0$. Let $\gamma = \inf\{t | h(0_1, \bar{F}(t)) = 0\}$.

Assume $\gamma > 0$. Then for all $0 < t < \gamma$, $h(0_1, \bar{F}(t)) > 0$ so that

$h(1_1, \bar{F}(t)) = h(0_1, \bar{F}(t))$ and for $t \geq \gamma$, $h(0_1, \bar{F}(t)) = 0$. Thus

$$e^{-\lambda t} = \begin{cases} h(0_1, \bar{F}(t)) = h(1_1, \bar{F}(t)) & \text{for } 0 < t < \gamma \\ \bar{F}_1(t) h(1_1, \bar{F}(t)) & \text{for } \gamma \leq t. \end{cases}$$

Now take $t_0 > \gamma$ and $0 < \alpha_0 < 1$ such that $\alpha_0 t_0 < \gamma$. It follows that

$$\begin{aligned} \exp(-\lambda \alpha_0 t_0) &= h(1_1, \bar{F}(\alpha_0 t_0)) \geq h(1_1, \bar{F}^{\alpha_0}(t_0)) \geq h^{\alpha_0}(1_1, \bar{F}(t_0)) \\ &\geq \bar{F}_1^{\alpha_0}(t_0) h^{\alpha_0}(1_1, \bar{F}(t_0)) = \exp(-\lambda \alpha_0 t_0). \end{aligned}$$

This gives that $\bar{F}_1(t_0) = 1$, a contradiction. Hence $\gamma = 0$ and so $h(0_1, \bar{F}(t)) = 0$ for all $t \geq 0$. Consequently for all $t \geq 0$,

$$e^{-\lambda t} = \bar{F}_1(t) h(1_1, \bar{F}(t)).$$

But for any $t \geq 0$ and $0 < \alpha < 1$,

$$e^{-\lambda \alpha t} = \bar{F}_1(\alpha t) h(1_1, \bar{F}(\alpha t)) \geq \bar{F}_1^\alpha(t) h^\alpha(1_1, \bar{F}(t)) = e^{-\lambda \alpha t}$$

and so $\bar{F}_1(\alpha t) = \bar{F}_1^\alpha(t)$. This gives that $\bar{F}_1(t) = e^{-\lambda_1 t}$, $t \geq 0$, for some $\lambda_1 > 0$ and so for all $t \geq 0$

$$\bar{F}(t) = e^{-\lambda t} = e^{-\lambda_1 t} h(1_1, \bar{F}(t))$$

which implies that $h(1_1, \bar{F}(t))$ is also exponential. But this is a monotonic system of order $(n-1)$ and so we can use the induction hypothesis.

(2.2a) Remark. As noticed by Esary and Marshall (1975), it follows immediately from the theorem that a multivariate exponential distribution whose one dimensional marginal distributions are lifetimes of coherent systems of independent IFRA distributions must be the multivariate distribution of Marshall and Olkin (1967).

(2.3) Example. Let X and Y be independent IFRA random variables and assume $\max(X,Y)$ is exponential. Then it follows that one of X or Y is exponential and the other has all its mass at zero.

The previous example is not peculiar in the sense that the assumption that $\max(X,Y)$ is exponential is prohibitively strong. This is illustrated in the following where X and Y are not assumed to be independent.

(2.4) Theorem. Let X, Y and $\max(X,Y)$ be exponential. Then $\min(X,Y)$ is exponential and $P\{X \leq Y\} = 1$ or $P\{X \geq Y\} = 1$. If X and Y are identically distributed or if the min and max are identically distributed, then $P\{X=Y\} = 1$.

Proof. Assume $X, Y, \max(X,Y)$ have means $\lambda_1^{-1}, \lambda_2^{-1}, (\lambda'_{12})^{-1}$. Thus $\exp(-\lambda_1 t) = P\{X > t\} \leq P\{\max(X,Y) > t\} = \exp(-\lambda'_{12} t)$ for all $t \geq 0$ and so $\lambda_1 \geq \lambda'_{12}$. Similarly $\lambda_2 \geq \lambda'_{12}$. Furthermore $0 \leq P\{\min(X,Y) > t\} = 1 - P\{X \leq t\} - P\{Y \leq t\} + P\{\max(X,Y) \leq t\} = \exp(-\lambda_1 t) + \exp(-\lambda_2 t) - \exp(-\lambda'_{12} t)$ and so $\exp(-\lambda_1 t) + \exp(-\lambda_2 t) \geq \exp(-\lambda'_{12} t)$ or $\exp(\lambda'_{12} - \lambda_1)t + \exp(\lambda'_{12} - \lambda_2)t \geq 1$. Now if $\lambda'_{12} < \lambda_1$ and $\lambda'_{12} < \lambda_2$ the above inequality fails for large t , hence $\lambda'_{12} = \lambda_1$ or $\lambda'_{12} = \lambda_2$. In the case $\lambda'_{12} = \lambda_1$, $P\{\min(X,Y) > t\} = \exp(-\lambda_2 t)$. This also implies that $P\{X \geq Y\} = 1$. The case $\lambda'_{12} = \lambda_2$ is similar. For the second part of the theorem assume λ_{12}^{-1} is the mean of $\min(X,Y)$. This yields

the identity $\exp(-\lambda_{12}t) + \exp(-\lambda'_{12}t) = \exp(-\lambda_1t) + \exp(-\lambda_2t)$ for all t . Clearly if $\lambda_1 = \lambda_2$ and then $\lambda'_{12} = \lambda_1 = \lambda_2$ and the above gives that $P\{X=Y\} = 1$. The case $\lambda_{12} = \lambda'_{12}$ is similar.

(2.5) Corollary. If $\max(aX, bY)$ is exponential for all $a, b \geq 0$, then $\min(aX, bY)$ is exponential for all $a, b \geq 0$.

Proof. Apply the above.

It turns out that an even stronger result holds than the one given in the corollary.

(2.6) Theorem. If $\max(aX, bY)$ is exponential for all $a, b \geq 0$, then there is a $c > 0$ such that $P\{X=cY\} = 1$.

Proof. It follows that X and Y are exponential. Let λ_1^{-1} and λ_2^{-1} be their means respectively. Choose $a, b > 0$ such that $\lambda_1 b = \lambda_2 a$. Then since aX and bY are identically distributed, we know from Theorem 2.4 that $aX = bY$ with probability one.

(2.7) Remark. Esary and Marshall (1974) have studied the class of distributions whose scaled minimums are exponential. The above theorem shows why this is fruitless for scaled maximums.

A result which is similar to Theorem 2.1 but deals with systems whose lifetime is the sum of its component lifetimes instead of monotonic systems is the following.

(2.8) Theorem. If the convolution of n IFRA distributions is exponential, then $(n-1)$ of the distributions are degenerate at zero and the other distribution is exponential.

Proof. It is sufficient to consider two independent IFRA random variables X and Y with survival functions \bar{F} and \bar{G} respectively such that for all $t \geq 0$ $P\{X+Y > t\} = e^{-\lambda t}$ with $\lambda > 0$. Since $0 < e^{-\lambda t} < 1$ for all $t > 0$, it easily follows that either $0 < \bar{F}(t) < 1$ for all $0 < t < \infty$ or $0 < \bar{G}(t) < 1$ for all $0 < t < \infty$. Assume the latter case holds. Suppose now that X is neither exponential nor degenerate at zero. Then there is some $0 < \alpha < 1$ and, by right-continuity, some interval $I \subset [0, \infty)$ such that $\bar{F}(\alpha x) > \bar{F}^\alpha(x)$ for all $x \in I$. Consequently for sufficiently large t , since \bar{G} is IFRA and $0 < \bar{G}(t) < 1$, we have that

$$e^{-\lambda \alpha t} = \int_0^\infty \bar{F}(\alpha t - y) dG(y) > \int_0^\infty \bar{F}^\alpha(t - y/\alpha) dG(y).$$

But

$$\int_0^\infty \{\bar{F}^\alpha(t - y/\alpha) dG(y)\} \geq \int_0^\infty \bar{F}(t - y) dG(y)^\alpha = e^{-\lambda \alpha t}$$

from Lemma 2.1 of Block and Savits (1976). Hence it must be that either X is exponential or degenerate at zero. Now if X is exponential, then $0 < \bar{F}(t) < 1$ for all $0 < t < \infty$ and the above argument can be repeated so that Y is either exponential or degenerate at zero. But since $0 < \bar{G}(t) < 1$ for all $t > 0$, it follows that Y must be exponential and so consequently $X + Y$ is gamma. This is a contradiction. Hence X must be degenerate at zero and therefore Y is exponential.

(2.9) Remark. An immediate corollary to the above is that a multivariate exponential distribution whose univariate marginal distributions are convolutions of independent IFRA distributions must have univariate marginals which are either pairwise independent or pairwise identical.

3. Applications.

In this section examples are constructed to show that certain definitions of multivariate IFRA distributions are not equivalent. We compare Condition C of Esary and Marshall (1975), the definition of multivariate IFRA (MIFRA) of Block and Savits (1977) and another definition designated Condition \sum . We list the definitions below.

(3.1) Definition.

1. (T_1, \dots, T_n) is MIFRA if

$$E[h(T_1, \dots, T_n)] \leq E^{1/\alpha}[h^\alpha(T_1/\alpha, \dots, T_n/\alpha)]$$

for all continuous nonnegative nondecreasing functions h and all $0 < \alpha \leq 1$.

2. (T_1, \dots, T_n) satisfies Condition C if for some independent IFRA random variables X_1, \dots, X_k and some coherent life functions τ_1, \dots, τ_n of order k , $T_i = \tau_i(X_1, \dots, X_k)$ for $i = 1, \dots, n$.

3. (T_1, \dots, T_n) satisfies Condition \sum if for some independent IFRA random variables X_1, \dots, X_k and some nonempty subsets $S_i \subset \{1, \dots, k\}$, $i = 1, \dots, n$,

$$T_i = \sum_{j \in S_i} X_j.$$

(3.2) Example (MIFRA \nRightarrow C). Consider $\bar{F}(x, y) = \exp(-\sqrt{x^2 + y^2})$ which has exponential marginals and is MIFRA. By Remark 2.2a if this distribution satisfied Condition C, it would have the Marshall and Olkin distribution, but it obviously does not.

(3.3) Example (MIFRA \nRightarrow $\{$). Let (T_1, T_2) be given by $T_1 = \min(X, Z)$ and $T_2 = \min(Y, Z)$, where X, Y and Z are independent exponential random variables. Then (T_1, T_2) is MIFRA. Now if (T_1, T_2) satisfied Condition $\{$, we would have by Remark 2.9 that either T_1 and T_2 are independent or that $T_1 = T_2$. However, both of these are impossible.

It should be mentioned that both Condition C and Condition $\{$ imply MIFRA as is shown in Block and Savits (1977).

References

1. Barlow, R. E. and Proschan, F. (1975). Statistical Theory of Reliability and Life Testing: Probability Models. Holt, Rinehart and Winston, New York.
2. Block, H. W. and Savits, T. H. (1976). The IFRA closure problem. Ann. Prob. 4, 1030-1032.
3. Block, H. W. and Savits, T. H. (1977). Multivariate IFRA Distributions. University of Pittsburgh Research Report #77-04.
4. Esary, J. D. and Marshall, A. W. (1974). Multivariate distributions with exponential minimums. Ann. Statistics 2, 84-96.
5. Esary, J. D. and Marshall, A. W. (1975). Multivariate IHRA distributions. Unpublished report.
6. Esary, J. D., Marshall, A. W. and Proschan, F. (1970). Some reliability applications of the hazard transform. SIAM J. Appl. Math. 18, 849-860.

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